Excess entropy in natural language: present state and perspectives

Łukasz Dębowski[1, a)]
Institute of Computer Science, Polish Academy of Sciences, Warszawa, Poland

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We review recent progress in understanding the meaning of mutual information in natural language. Let us define words in a text as strings that occur sufficiently often. In a few previous papers, we have shown that a power-law distribution for so defined words (a.k.a. Herdan's law) is obeyed if there is a similar power-law growth of (algorithmic) mutual information between adjacent portions of texts of increasing length. Moreover, the power-law growth of information holds if texts describe a complicated infinite (algorithmically) random object in a highly repetitive way, according to an analogous power-law distribution. The described object may be immutable (like a mathematical or physical constant) or may evolve slowly in time (like cultural heritage). Here we reflect on the respective mathematical results in a less technical way. We also discuss feasibility of deciding to what extent these results apply to the actual human communication.

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In 1990, German engineer Wolfgang Hilberg published an article[1] where the graph of conditional entropy of printed English from Claude Shannon's famous work[2] was replotted in log-log scale. Seeing a dozen data points lie on a straightish line, he conjectured that entropy of a block of n characters drawn from a text in natural language is roughly proportional to $\sqrt{n}$ for n tending to infinity. Although this conjecture was not sufficiently supported by experiment or a rational model, it attracted interest of a few physicists seeking to understand complex systems.[3–6] As a graduate in physics and a junior computational linguist, I found their publications in 2000. They stimulated me to ponder upon the interplay of randomness, order, and complexity in language. I felt that better understanding of Hilberg's conjecture can lead to better understanding of Zipf's law for the distribution of words.[7,8] Using Hilberg's conjecture, I wished to demonstrate clearly that the monkey-typing model, introduced to explain Zipf's law,[9] cannot account for some important purposes of human communication. However, it took a few years to translate these intuitions into a mature mathematical model.[10–13] The model is presented here in an accessible way. I also identify a few problems for future research.

I. INTRODUCTION

The phenomenon of human language communication can be looked upon from various perspectives. Respectively, these different points of view give rise to different mathematical models, which are applied to human language, studied for themselves, or used for different purposes. The most clear dichotomy of mathematical views onto language comes from whether we look at individual sentences or above.

On the one hand, we may ask how human beings understand individual sentences and what rules are obeyed in their composition. This interest leads to elaborate theories of phonology, word morphology, syntax, automata, formal and programming languages, mathematical logic and formal semantics.[14–16] Although fragmented, these fields influence one another. Their common feature is using discrete rather than numerical models. Thus, they may be called non-quantitative linguistics (non-QL).

On the other hand, we may ask how sentences are chained into texts, discourses, or collections of texts typically produced by humans. At this level, rigid structures are less prominent and quantitative analysis of data, done under auspices of quantitative linguistics (QL) or corpus linguistics, forms the primary tool of description. However, in spite of a few remarkable observations like Zipf's[7] or Menzerath's[17] laws, QL has not established a coherent mathematical framework so far.[18]

Although communication between QL and non-QL is weak because of using very different mathematical notions, quantitative reflection upon language and difficulties of probabilistic modeling thereof inspired a few great mathematicians: A. Markov formulating the notion of a Markov chain,[19] C. Shannon establishing information theory,[2,20] B. Mandelbrot studying fractals,[8,21] and A. Kolmogorov introducing algorithmic complexity.[22]

In this paper, I present some conceptual framework for QL which borrows heavily from information theory and yields a macroscopic view onto human communication. Because of the exposed connections among mutual information, power laws, and emergence of hierarchical patterns in data, I suppose that my results may be interesting for researchers in the domain of complex systems, who consider the power-law growth of mutual informa-

a) ldebowsk@ipipan.waw.pl; www.ipipan.waw.pl/~ldebowsk;

tion a hallmark of complex behavior.[5,6,23] How to combine the 'macroscopic' QL and the 'microscopic' non-QL into a larger theory of language is a different problem. I consider it worth pursuing but harder.

The central point of my paper is linking Herdan's law, an empirical power law for the number of different words,[24–27] with an intuitive idea that texts describe various facts in a highly repetitive and mostly logically consistent way. Thus I will discuss a proposition that can be expressed informally as follows:

(H) If a text of length n describes n^β independent facts in a repetitive way, where $\beta \in (0, 1)$, then the text contains at least $n^\beta / \log n$ different words.

Proposition (H) has been formalized and proved by myself in a series of mathematical definitions and theorems.[11–13] It holds under an appropriate quantification over n, which is a combination of an upper and a lower limit over n.

Let me note that Proposition (H) can be also linked to the relaxed Hilberg hypothesis. This conjecture says that (algorithmic) mutual information between adjacent blocks of text of length n is roughly proportional to n^β.[1,3–6,23] Besides Proposition (H), I have formalized and proved the following two propositions:

(H') If a text of length n describes n^β independent facts in a repetitive way, where $\beta \in (0, 1)$, then mutual information between adjacent blocks of length n exceeds n^β.

and:

(H") If mutual information between adjacent blocks of n of length n exceeds n^β, where $\beta \in (0, 1)$, then the text of length n contains at least $n^\beta / \log n$ different words.

The quantifications over n in the formalizations of Propositions (H') and (H") are analogical as in the Proposition (H). For this reason Proposition (H) does not follow from the conjunction of Propositions (H') and (H"). All these propositions are, however, true. The significance of the propositions is as follows. On the one hand, Proposition (H') demonstates that Hilberg's hypothesis can be motivated rationally. On the other hand, Proposition (H") shows that the hypothesis implies certain empirical regularities, such as Herdan's law, even if there are problems with verifying Hilberg's conjecture directly.

Consecutively, I will introduce the concepts that appear in Propositions (H), (H'), and (H") and their formal statements. I will also discuss some related problems. The composition of the paper is as follows: In Section II, I introduce the motivating linguistic concepts. In Section III, I discuss the mathematical results. In Section IV, I reflect upon limitations of these results as a theory of human language or other complex communication systems. Section V contains important remarks for researchers wishing to verify Hilberg's hypothesis experimentally. Section VI concludes the paper.

II. IDEAS IN THE BACKGROUND

Before we embark on discussing formal models, I should introduce some linguistic playground on which the models will be built. First, I will recall empirical laws for the distribution of words. Second, I will introduce grammar-based codes as a method of detecting word boundaries. Third, Hilberg's hypothesis and its generalizations will be presented. In the end, I will discuss the idea of texts that describe infinitely many facts in a highly repeatable and logically consistent way.

A. Zipf's and Herdan's laws

A few famous empirical laws of quantitative linguistics concern the distribution of words. Amongst them, the Zipf-(Mandelbrot) law is the most celebrated.[7,8] According to this law, the word frequency $f(w)$ in a text is an inverse power of the word rank $r(w)$, i.e.,

$$f(w) \propto \left[\frac{1}{B + r(w)} \right]^{1/\beta}. \tag{1}$$

The frequency $f(w)$ of word w is defined as the number of its occurrences in the text, whereas the word rank $r(w)$ is the position of w on the list of words sorted by decreasing frequencies. Constant B is positive whereas constant $\beta \in (0, 1)$ is close to 1 for $r(w) \lesssim 10^3 \div 10^4$. For larger ranks this relationship is breaks down and β can drop much closer to 0, depending on the text composition.[28,29]

Zipf's law attracts attention of many theoreticians wishing to explain it. The most famous explanation of Zipf's law is given by the 'monkey-typing' model. In this explanation, the text is assumed to be a sequence of independent identically distributed (IID) variables taking values of both letters and spaces and, as a result, the Zipf-Mandelbrot law is satisfied for strings of letters delimited by spaces[8,9]. Other known explanations involve, e.g., multiplicative processes[30,31] games,[32] and information theoretic arguments.[33–36]

In this paper, we will focus on a certain corollary of the Zipf-Mandelbrot law, namely a relationship between the length of the text and the number of different words therein. This relationship is usually called Herdan's or Heaps' law in the English literature.[24–27] It takes form of an approximate empirical power law

$$V \propto n^\beta, \tag{2}$$

where V is the number of different words and n is the text length (in characters). We can see that (2), up to a multiplicative logarithmic term, appears in the conclusion of Propositions (H) and (H").

The Herdan-Heaps law can be inferred from the Zipf-Mandelbrot law assuming certain regularity of text growth.[37,38] In particular, if law (1) were satisfied exactly then (2) would hold automatically. We have the following proposition:

Proposition 1 *Let N be the number of all words in the text and V be the number of different words. If (1) is satisfied with $B = 0$, β constant, and $f(w)/N$ constant w.r.t. N for the most frequent word w then we have $V \propto N^{\beta}$.*[38]

Proof: For the least frequent word u we have the frequency $f(u) = 1 \propto V^{1/\beta}$. Hence the proportionality constant equals $V^{1/\beta}$. Thus for the most frequent word w we have $f(w) = V^{1/\beta}$. Because $f(w)/N$ was assumed constant, we obtain $V \propto N^{\beta}$. $\square$

In reality it happens that relationships (1) and (2) are quite inexact and the best fit for (2) yields a β smaller than for (1).

In this article, I propose another explanation of Herdan's law, which is probabilistic. In any such explanation, two postulates are adopted more or less explicitly. The first postulate concerns how words are delimited in the text. The second postulate concerns what kind of stochastic process is suitable for modeling the text. My explanation of Herdan's law targets two modeling challenges:

1. Words can be delimited in the text even when the spaces are absent.
2. Texts refer to many facts unknown a priori to the reader but they usually do this in a consistent and repetitive way.

The necessary notions will be explained consecutively. First, we will revisit the concept of a word. Second, we will address the properties of texts.

B. Detecting word boundaries with grammar-based codes

In this section, we will discuss how to delimit words in a text and, consecutively, how to count their number. If we agree that texts are sequences of characters taking values of both letters and delimiters (such as spaces), the most obvious choice, suggested by orthographies of many languages, is to define words as strings of letters separated by delimiters. There are, however, kinds of texts or languages where words are not separated by delimiters on a regular basis (ancient Greek, modern Chinese, or spoken English as a speech signal).

Seeking for an absolute criterion for word boundaries, linguists observed that strings of characters that are repeated within the text significantly many times often correspond to whole words or set phrases (multi-word expressions) like *United States*.[39,40] Another important insight is that the number of so detected 'words' or 'phrases' is a thousand times larger in texts produced by humans than in texts generated by IID sources.[41]

A particularly convenient way to detect words or sufficiently often repeated strings is to use a grammar-based code that minimizes the length of a certain text encoding.[42,43] Grammar-based codes compress strings by transforming them first into special grammars, called admissible grammars,[44] and then encoding the grammars back into strings according to a fixed simple method. An admissible grammar is a context-free grammar that generates a singleton language $\{w\}$ for some string w.[44]

In an admissible grammar there is exactly one rule per nonterminal symbol and the nonterminals can be ordered so that the symbols are rewritten onto strings of strictly succeeding symbols.[44,45] A particular example of an admissible grammar is as follows,

$$\left\{ \begin{array}{ll} A_1 \rightarrow & \textbf{How\_much\_}A_5\textbf{\_w}A_4A_2A_3\textbf{,} \\ & \textbf{if}A_2\textbf{c}A_4\textbf{\_}A_3\textbf{\_}A_5\textbf{?} \\ A_2 \rightarrow & \textbf{\_a\_}A_5A_3\textbf{\_} \\ A_3 \rightarrow & \textbf{chuck} \\ A_4 \rightarrow & \textbf{ould} \\ A_5 \rightarrow & \textbf{wood} \end{array} \right\},$$

where A_1 is the initial symbol and other A_i are secondary nonterminal symbols. If we start the derivation with symbol A_1 and follow the rewriting rules, we obtain the text of a verse:

How much wood
would a woodchuck chuck,
if a woodchuck
could chuck wood?

Although it cannot be seen in the short text above, secondary nonterminals A_i often correspond to words or set phrases in compressions of longer texts. This correspondence is particularly good if it is additionally required that nonterminals are defined as strings of only terminal symbols.[43] For this reason, the number of different words in an arbitrary text will be modeled in the formalization of Propositions (H) and (H") by the number of different nonterminals in a certain admissible grammar.

C. Excess entropy and Hilberg's hypothesis

Once I have partly described how to detect and count "words" in an arbitrary text, let us refine our ideas about texts typically produced by humans. There are justified opinions that such texts result from a very complicated amalgam of deterministic computation and randomness[46] and this amalgam can be realized very differently in particular texts, as mocked by D. Knuth.[47] To make these intuitions more precise, let us investigate entropy and algorithmic complexity of texts.

Let us begin with entropy. For a probability space $(\Omega, \mathfrak{J}, P)$, the entropy of a discrete random variable X is defined as

$$H_P(X) := -\mathbf{E}_P \log P(X), \tag{3}$$

where $\mathbf{E}_P$ is the expectation with respect to P and random variable $P(X)$ takes value $P(X = x)$ for $X = x$.

Subsequently, for a discrete stationary process $(X_i)_{i\in\mathbb{Z}}$, we define n-symbol entropy

$$H_\mu(n) := H_P(X_1^n), \tag{4}$$

where $X_n^m = (X_i)_{n\le i\le m}$ are blocks of variables and $\mu = P((X_i)_{i\in\mathbb{Z}} \in \cdot)$ denotes the distribution of $(X_i)_{i\in\mathbb{Z}}$ (i.e., $\mu(A) = P((X_i)_{i\in\mathbb{Z}} \in A)$). On the one hand, if the process is purely random, i.e., X_i are IID variables, then $H_\mu(n) \propto n$. On the other hand, we have $H_\mu(n) = \text{const}$ if the process is in a sense deterministic, i.e., $X_i = f(X_1^{i-1})$.

Intuitively, texts written by humans are neither deterministic nor purely random. This corresponds to a particular behavior of entropy $H_\mu(n)$. Some insight into this behavior can be obtained by asking people to guess the next character of a text given the context of n previous characters. In one of his very first papers on information theory,[2] Shannon performed this experiment. As it was later observed by Hilberg,[1] Shannon's data points obey approximate relationship

$$H_\mu(n) \propto n^\beta \tag{5}$$

for $\beta \approx \frac{1}{2}$, $n \lesssim 100$, and $H_\mu(n)$ being an estimate of entropy of n consecutive characters rather than the entropy itself. Hilberg supposed that (5) also holds for much larger n, even for n tending to infinity.

Some parallel research in entropy of texts in natural language suggests that estimates of entropy depend heavily on a particular text[48,49] and Shannon's guessing method does not give precise estimates of entropy for large n.[50] Thus Hilberg's conjecture (5) should be modified and other ways of its justification should be sought.

First of all, let us recall the concept of entropy rate

$$h_\mu := \lim_{n\to\infty} \frac{H_\mu(n)}{n}. \tag{6}$$

For a stationary process, conjecture (5) implies entropy rate $h_\mu = 0$, which is equivalent to asymptotic determinism, i.e., $X_1 = f((X_i)_{i<1})$ almost surely. Such asymptotic determinism seems an unrealistic assumption. (But we may be wrong.)

Thus let us introduce block mutual information

$$\begin{aligned} E_\mu(n) &:= I_P(X_1^n; X_{n+1}^{2n}) \\ &:= H_P(X_1^n) + H_P(X_{n+1}^{2n}) - H_P(X_1^{2n}), \end{aligned} \tag{7}$$

called n-symbol excess entropy.[6] $E_\mu(n)$ is a convenient measure of complexity of discrete-valued processes. It vanishes for purely random processes and is bounded for asymptotically deterministic ones. Now let us observe that for a stationary process $(X_i)_{i\in\mathbb{Z}}$, we have $H_P(X_{n+1}^{2n}) = H_P(X_1^n)$ and we obtain

$$E_\mu(n) \propto n^\beta \tag{8}$$

if (5) is satisfied. We will call (8) the relaxed Hilberg conjecture. Notice that, unlike the case of (5), $h_\mu = 0$ does not follow from (8).

Thus, if proportionality (8) were actually satisfied for any n then texts in natural language could not be produced by generalized 'monkey-typing'. In the generalized 'monkey-typing' model, the text is generated by a finite-state source a.k.a. a hidden Markov model. Indeed, if the finite-state source has k hidden states then $E_\mu(n) \le \log k$.[51]

Nonetheless, relationship (8) does not exhaust the problem of reasonable generalizations. Bluntly speaking, it seems impossible to point out a correct reference measure P for texts in natural language.[22] Although researchers in linguistics happen to speak of entropies of a single text,[48,49] this is an abuse of concepts because entropy is a function of a distribution rather than of a text! To render the relaxed Hilberg conjecture for an individual text x_1^n, we should use prefix algorithmic complexity $H(x_1^n)$ instead of entropy $H_P(X_1^n)$. Formally, prefix complexity $H(x_1^n)$ is defined as the length of the shortest self-delimiting program to generate text x_1^n.[52]

Thus for algorithmic mutual information

$$I(x_1^n; x_{n+1}^{2n}) := H(x_1^n) + H(x_{n+1}^{2n}) - H(x_1^{2n}), \tag{9}$$

we will call relationship

$$I(x_1^n; x_{n+1}^{2n}) \propto n^\beta \tag{10}$$

the relaxed Hilberg conjecture for individual texts. This relationship makes quite a sense because in the probabilistic setting we have

$$H_P(X_1^n) \le \mathbf{E}_P\, H(X_1^n) \le H_P(X_1^n) + C_n^P \tag{11}$$

for any computable measure P and $C_n^P = c_P + 2\log n$ with $c_P < \infty$.[53] We remind that measure P is called computable when $P(X_1^n)$ can be computed given X_1^n by a fixed Turing machine. Under this assumption, law (8) follows up to a logarithmic correction if proportionality (10) holds almost surely for a fixed proportionality constant.

D. Highly repetitive descriptions of a random world

In this subsection, I want to discuss the question *why* texts typically produced by humans diverge from both simple randomness and determinism. This will provide a justification for Hilberg's conjecture. I may point out three plausible reasons:

A. Texts attempt to describe an infinite collection of independent facts that concern either an immutable objective reality or an evolving historical heritage.

B. For some reasons, the immutable objective reality and the historical heritage are described in a highly repetitive and mostly logically consistent way.

C. Any fact about the immutable objective reality can be inferred correctly given sufficiently many texts, according to a fixed inference method, regardless of where we start reading.

As I will show in Subsection III A, the conjunction of propositions A.–C. implies Hilberg's conjecture by a formalization of Proposition (H'). Thus let us inspect these statements closer.

As for postulate A., there exists a collection of facts about an immutable objective reality which is infinite and algorithmically random. A particular collection of that kind is given by the binary expansion of halting probability Ω. The expansion of Ω is an algorithmically random sequence and represents a large body of mathematical knowledge in its most condensed form.[52,54] (Sequence $(x_i)_{i\in\mathbb{N}}$ is called algorithmically random for algorithmic complexity $H(x_1^n) \gtrsim n$.) Other plausible choices of immutable and algorithmically random sequences are binary expansions of compressed physical constants.

In contrast, the evolving historical heritage, which is primarily described in texts, admits a larger interpretation. Namely, this heritage encompasses both the culture and the present state of the physical world. We can also agree that the present state of the physical world contains all material aspects of the culture.

To make these simple statements less abstract, let us mention a few examples of what falls under the evolving historical heritage. The scope of culture covers: vocabulary and grammars of particular languages, fictitious worlds described in novels, all heritage of arts, humanities, science, and engineering. The present state of physical world covers also all facts of biology, geography, and astronomy, including those yet unknown.

To support postulate B., let us consider why the facts mentioned in texts are described in a highly repetitive and mostly logically consistent way. This has more to do with the human nature than with properties of the described world itself. As a plausible reason, I suppose that human society develops communication structures to maintain a larger body of knowledge than any individual could manage on his or her own.

Thus the the primary cause of repetition is probably the requirement that knowledge is passed from generation to generation. Moreover, I suppose that any human mind needs constant restimulation to remember and reorganize the possessed knowledge. The result is that either in fiction or in scientific writings, people prefer logically consistent and directed narrations. This consistency also implies repetition.

To argue in favor of postulate C., let us observe the following. In the course of time, the historical heritage undergoes distributed creation, accumulation, description, and lossy transmission from text creators to text addressees. This should be contrasted with the immutable objective reality, which can be discovered and described independently by successive generations of text creators.

Thus it does not sound weird that every fact about the immutable objective reality is described in some text ultimately and repeated infinitely many times afterwards. Moreover, there should exist a fixed method of interpreting texts in natural language to infer these facts. Such faculty is called human language competence in the linguistic jargon and it allows knowledge to be passed from generation to generation.

III. MATHEMATICAL SYNTHESIS

The ideas presented in the previous section will now be synthesized as an assortment of theorems and toy examples of stochastic processes. This can be called a formalization of Propositions (H), (H') and (H"), mentioned in the Introduction. Namely, in a series of theorems I will link Hilberg's conjecture with Herdan's law for vocabulary size of admissible grammars and a power law for the number of facts that can be inferred from a given text. Afterwards, I will demonstrate a few simple processes that exhibit all three laws. For simplicity of argumentation, I will discuss probabilistic Hilberg hypothesis (8) rather than algorithmic one (10). Respectively, both texts and facts will be modeled by random variables.

In the following, symbol $\mathbb{N}$ denotes the set of positive integers. For a countable alphabet $\mathbb{X}$, the set of nonempty strings is $\mathbb{X}^+ := \bigcup_{n\in\mathbb{N}} \mathbb{X}^n$ and the set of all strings is $\mathbb{X}^* := \mathbb{X}^+ \cup \{\lambda\}$, where λ stands for the empty string. The length of a string $w \in \mathbb{X}^*$ is written as $|w|$.

A. Definitions and theorems

In this subsection I will show how Proposition (H) can be formalized. First, the model of texts and facts is made precise. Second, the model of words is elaborated. Third, I present three previously proved theorems[12] that link Hilberg's conjecture and these two models.

Let $(X_i)_{i\in\mathbb{Z}}$ be a discrete stochastic process with variables $X_i : \Omega \to \mathbb{X}$, where Ω denotes the event space. Process $(X_i)_{i\in\mathbb{Z}}$ models an infinite text, where X_i are characters if $\mathbb{X}$ is finite or sentences if $\mathbb{X}$ is infinite. Moreover, let $Z_k : \Omega \to \{0,1\}$, where $k \in \mathbb{N}$, be equidistributed IID binary variables. Variables Z_k model facts described in text. Their values (1=true and 0=false) can be interpreted as logical values of certain systematically enumerated independent propositions.

More specifically, let us assume that each fact Z_k can be inferred from a half-infinite text according to a fixed method if we start reading it from an arbitrary position, like in postulate C. from Subsection II D. The method to infer these facts will be formalized as certain functions s_k which given a text predict whether the k-th fact is true or false. This leads to the following definition.

Definition 1 *A stochastic process $(X_i)_{i\in\mathbb{Z}}$ is called strongly nonergodic if there exists an IID binary process $(Z_k)_{k\in\mathbb{N}}$ with marginal distribution*

$$P(Z_k = 0) = P(Z_k = 1) = \frac{1}{2} \tag{12}$$

and functions $s_k : \mathbb{X}^* \to \{0,1\}$, where $k \in \mathbb{N}$, such that

$$\lim_{n\to\infty} P(s_k(X_{t+1}^{t+n}) = Z_k) = 1, \quad \forall t \in \mathbb{Z}, \forall k \in \mathbb{N}. \tag{13}$$

In the definition above, facts Z_k are fixed for a given realization of $(X_i)_{i\in\mathbb{Z}}$ but they can be very different for different realizations. I suppose that such probabilistic modeling of both texts and facts reflects some properties of language, where reality described in texts is most often created at random during text generation and recalled afterwards. Under this assumption I will derive an average-case result.

Strong nonergodicity is indeed a stronger condition than nonergodicity. A stationary process is strongly nonergodic when there exists a continuous random variable $\Theta : \Omega \to (0,1)$ measurable with respect to the shift-invariant algebra.[10] Such a variable is an example of a parameter in terms of Bayesian statistics. Taking $\Theta = \sum_{k=1}^{\infty} 2^{-k} Z_k$ corresponds to a uniform prior distribution on Θ.

The number of facts described in text X_1^n will be identified with the number of Z_k's that may be predicted with probability greater than δ given X_1^n. That is, this number is understood as the cardinality $\operatorname{card} U_\delta(n)$ of set

$$U_\delta(n) := \{k \in \mathbb{N} : P\left(s_k\left(X_1^n\right) = Z_k\right) \geq \delta\}. \tag{14}$$

There is also another condition for process $(X_i)_{i\in\mathbb{Z}}$, which is stronger than requiring entropy rate $h > 0$.

Definition 2 *A process $(X_i)_{i\in\mathbb{Z}}$ is called a finite-energy process if*

$$P(X_{t+|w|+1}^{t+|wu|} = u | X_{t+1}^{t+|w|} = w) \leq K c^{|u|}$$

for all $t \in \mathbb{Z}$, all $u, w \in \mathbb{X}^*$, and certain constants $c < 1$ and K, as long as $P(X_{t+1}^{t+|w|} = w) > 0$.

The term "finite-energy process" has been coined by Shields.[55] We are unaware of the motivation for this name.

Now let us discuss the adopted model of words. It uses admissible grammars mentioned in Subsection II B. A function Γ such that $\Gamma(w)$ is a grammar and generates language $\{w\}$ for each string $w \in \mathbb{X}^+$ is called a grammar transform.[44] Any such grammar $\Gamma(w)$ is admissible and is given by its set of production rules

$$\Gamma(w) = \left\{ \begin{array}{l} A_1 \to \alpha_1, \\ A_2 \to \alpha_2, \\ ..., \\ A_n \to \alpha_n \end{array} \right\}, \tag{15}$$

where A_1 is the start symbol, other A_i are secondary nonterminals, and the right-hand sides of rules satisfy $\alpha_i \in (\{A_{i+1}, A_{i+2}, ..., A_n\} \cup \mathbb{X})^*$. The number of distinct nonterminal symbols in grammar (15) will be called the vocabulary size of $\Gamma(w)$ and denoted by

$$\mathbf{V}[\Gamma(w)] := \operatorname{card}\{A_1, A_2, ..., A_n\} = n. \tag{16}$$

In the following, let us consider vocabulary size of admissibly minimal grammar transforms, which were defined exactly in the previous paper.[12] The formal definition is too long to quote here but, briefly speaking, admissibly minimal grammar transforms minimize a certain nice length function of grammars. A simple example of a grammar length function is Yang-Kieffer length

$$|\Gamma(w)| := \sum_{i=1}^{n} |\alpha_i| \tag{17}$$

for grammar (15), where $|\alpha_i|$ is the length of the right-hand side of rule $A_i \to \alpha_i$.[45]

In our application we use a slightly different length function $||\Gamma(w)||$, which measures the length of $\Gamma(w)$ after a certain reversible binary encoding, and we choose a grammar transform that minimizes $||\Gamma(w)||$ for a given string w. Nonterminals of these so called admissibly minimal grammar transforms often correspond to words in the linguistic sense.[42,43] Thus we stipulate that the vocabulary size of an admissibly minimal grammar is close to the number of distinct words in the text.

The formalization of Proposition (H) is as follows:

Theorem 1 *Let $(X_i)_{i\in\mathbb{Z}}$ be a stationary finite-energy strongly nonergodic process over a finite alphabet $\mathbb{X}$. If*

$$\liminf_{n\to\infty} \frac{\operatorname{card} U_\delta(n)}{n^\beta} > 0 \tag{18}$$

holds for some $\beta \in (0,1)$ and $\delta \in (\frac{1}{2}, 1)$ then

$$\limsup_{n\to\infty} \mathbf{E}_P \left(\frac{\mathbf{V}[\Gamma(X_1^n)]}{n^\beta (\log n)^{-1}} \right)^p > 0, \quad p > 1, \tag{19}$$

for any admissibly minimal grammar transform Γ.

There are also two similar theorems that link inequalities (18) and (19) with Hilberg's conjecture. These are formalizations of Propositions (H') and (H") respectively.

Theorem 2 *Let $(X_i)_{i\in\mathbb{Z}}$ be a stationary strongly nonergodic process over a finite alphabet $\mathbb{X}$. If (18) holds for some $\beta \in (0,1)$ and $\delta \in (\frac{1}{2}, 1)$ then we have*

$$\limsup_{n\to\infty} \frac{E_\mu(n)}{n^\beta} > 0. \tag{20}$$

Theorem 3 *Let $(X_i)_{i\in\mathbb{Z}}$ be a stationary finite-energy process over a finite alphabet $\mathbb{X}$. Assume that*

$$\liminf_{n\to\infty} \frac{E_\mu(n)}{n^\beta} > 0 \tag{21}$$

holds for some $\beta \in (0,1)$. Then we have (19) for any admissibly minimal grammar transform Γ.

Theorem 1 does not follow from Theorems 2 and 3 because (20) is a weaker condition than (21). However, all these propositions are true and the proofs of these propositions are almost simultaneous.[12] By an easy argument, using Lemma 1 from Subsection V B, it can be also shown that n-symbol excess entropy $E(n)$ in Theorems 1–3 may be replaced with expected algorithmic information $\mathbf{E}_P\, I(X_1^n; X_{n+1}^{2n})$.

B. The zoo of Santa Fe processes

Now I will present a few stochastic processes to which my theorems may be applied.[10–13] These processes are merely simple mathematical models that satisfy hypotheses of Theorems 1, 2, and 3. They model some aspects of human communication but they do not pretend to be very realistic models of language. The purpose of these constructions is to enhance our imagination and to show that the hypotheses of the theorems can be satisfied.

Quite early in my investigations I came across the following process. Let the alphabet be $\mathbb{X} = \mathbb{N} \times \{0,1\}$ and let the process $(X_i)_{i\in\mathbb{Z}}$ have the form

$$X_i := (K_i, Z_{K_i}), \tag{22}$$

where $(Z_k)_{k\in\mathbb{N}}$ and $(K_i)_{i\in\mathbb{Z}}$ are probabilistically independent. Moreover, let Z_k be IID with marginal distribution (12) and let $(K_i)_{i\in\mathbb{Z}}$ be such an ergodic stationary process that $P(K_i = k) > 0$ for every natural number $k \in \mathbb{N}$. Under these assumptions it can be demonstrated that $(X_i)_{i\in\mathbb{Z}}$ forms a strongly nonergodic process.[10] I will call process $(X_i)_{i\in\mathbb{Z}}$ with variables X_i as in (22) the Santa Fe process because I discovered it during my visit to the Santa Fe Institute.

Santa Fe process (22) can be interpreted as a sequence of statements which describe a fixed random object $(Z_k)_{k\in\mathbb{N}}$ in a repetitive and consistent way. Each statement $X_i = (k, z)$ reveals both the address k of a random bit of $(Z_k)_{k\in\mathbb{N}}$ and its value $Z_k = z$. The description is consistent, namely, if two statements $X_i = (k, z)$ and $X_j = (k', z')$ describe the same bits ($k = k'$) then they always assert identical value ($z = z'$).

Moreover, we can see that the revelation of the bit address is important to assure the existence of functions s_k such that (13) holds. Indeed we may take

$$s_k(v) := \begin{cases} 0 & \text{if } (k,0) \sqsubseteq v \text{ and } (k,1) \not\sqsubseteq v, \\ 1 & \text{if } (k,1) \sqsubseteq v \text{ and } (k,0) \not\sqsubseteq v, \\ 2 & \text{else,} \end{cases} \tag{23}$$

where we write $u \sqsubseteq v$ when a sequence v contains string u as a substring.

For these functions s_k, I have shown[12] that the cardinality of set $U_\delta(n)$ obeys

$$\operatorname{card} U_\delta(n) \geq \left[\frac{n}{-\zeta(\beta^{-1}) \log(1-\delta)} \right]^\beta \tag{24}$$

for process (22) if variables K_i are IID and power-law distributed,

$$P(K_i = k) = k^{-1/\beta}/\zeta(\beta^{-1}), \qquad \beta \in (0,1), \tag{25}$$

where $\zeta(\alpha) = \sum_{k=1}^\infty k^{-\alpha}$ is the zeta function.

In contrast, it can be seen that the cardinality of set $U_\delta(n)$ is of order $\log n$ if $(X_i)_{i\in\mathbb{Z}}$ is a Bernoulli process with binary variables $X_i : \Omega \to \{0,1\}$, a random parameter $\Theta = \sum_{k=1}^\infty 2^{-k} Z_k$, and conditional distribution

$$P(X_1^n || \Theta) = \prod_{i=1}^n \Theta^{X_i} (1-\Theta)^{1-X_i}. \tag{26}$$

Next, let us discuss a certain modification of the Santa Fe process. As I have said before, facts that are mentioned in texts repeatedly fall roughly under two types: (a) facts about objects that do not change in time (like mathematical or physical constants), and (b) facts about objects that evolve with a varied speed (like culture, language, or geography). An attempt to model the latter phenomenon leads to processes that are mixing, as we will see now.

In the following, let us replace individual variables Z_k in the Santa Fe process with Markov chains $(Z_{ik})_{i\in\mathbb{Z}}$. The Markov chains are formed by iterating a binary symmetric channel. Consecutively, let us put

$$X_i = (K_i, Z_{i,K_i}), \tag{27}$$

where processes $(K_i)_{i\in\mathbb{Z}}$ and $(Z_{ik})_{i\in\mathbb{Z}}$, where $k \in \mathbb{N}$, are independent and distributed as follows. First, variables K_i are distributed according to formula (25), as before. Second, each process $(Z_{ik})_{i\in\mathbb{Z}}$ is a Markov chain with marginal distribution

$$P(Z_{ik} = 0) = P(Z_{ik} = 1) = \frac{1}{2} \tag{28}$$

and cross-over probabilities

$$P(Z_{ik} = z | Z_{i-1,k} = 1 - z) = p_k, \quad z \in \{0,1\}. \tag{29}$$

The random object $(Z_k)_{k\in\mathbb{N}}$ described by original Santa Fe process (22) does not evolve, or rather, no bit Z_k is ever forgotten once revealed. In contrast, the random object $(Z_{ik})_{k\in\mathbb{N}}$ described by modified Santa Fe process (27) is a function of time i and the probability that the k-th bit flips at a given instant equals p_k. For $p_k = 0$, process (27) collapses to process (22).

As I have shown previously,[13] the modified Santa Fe process defined in (27) is mixing for $p_k \in (0,1)$, and thus ergodic. Moreover, for $p_k \in [0,1]$, I have also demonstrated asymptotics

$$\lim_{n\to\infty} \frac{E_\mu(n)}{n^\beta} = \frac{(2-2^\beta)\Gamma(1-\beta)}{[\zeta(\beta^{-1})]^\beta} \tag{30}$$

if $\lim_{k\to\infty} p_k / P(K_i = k) = 0$ and K_i obey law (25).[13] In the equation above $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ is the gamma function. Formula (30) follows from approximating an exact expression for $E_\mu(n)$ with an integral. Note that (30) holds also in the case of original strongly nonergodic Santa Fe process (22).

Neither of processes defined so far is a process over a finite alphabet, as required in Theorems 1–3. To construct the desired processes over a ternary alphabet, I have used stationary (variable length) coding of

processes over one alphabet into processes over another alphabet. This transformation preserves stationarity, (non)ergodicity, and entropy—to some extent.[11,13] Despite elaborate notation, the idea of this transformation is quite simple.

First, let a function $f : \mathbb{X} \to \mathbb{Y}^*$, called a coding function, map symbols from alphabet $\mathbb{X}$ into strings over another alphabet $\mathbb{Y}$. We define its extension to double infinite sequences $f^{\mathbb{Z}} : \mathbb{X}^{\mathbb{Z}} \to \mathbb{Y}^{\mathbb{Z}} \cup (\mathbb{Y}^* \times \mathbb{Y}^*)$ as

$$f^{\mathbb{Z}}((x_i)_{i\in\mathbb{Z}}) := ...f(x_{-1})f(x_0)\boldsymbol{.}f(x_1)f(x_2)..., \tag{31}$$

where $x_i \in \mathbb{X}$ and the bold-face dot separates the 0-th and the first symbol. Then for a stationary process $(X_i)_{i\in\mathbb{Z}}$ with variables $X_i : \Omega \to \mathbb{X}$, we define process

$$(Y_i)_{i\in\mathbb{Z}} := f^{\mathbb{Z}}((X_i)_{i\in\mathbb{Z}}) \tag{32}$$

with variables $Y_i : \Omega \to \mathbb{Y}$.

In the following application, let us assume the infinite alphabet $\mathbb{X} = \mathbb{N} \times \{0, 1\}$, the ternary alphabet $\mathbb{Y} = \{0, 1, 2\}$, and the coding function

$$f(k, z) = b(k)z2, \tag{33}$$

where $b(k) \in \{0, 1\}^+$ is the binary representation of a natural number k stripped of the leading digit 1.

Transformation (32) does not preserve stationarity in general but process $(Y_i)_{i\in\mathbb{Z}}$ is asymptotically mean stationary (AMS) for process (27) and coding function (33).[11] Then for the distribution $\nu = P((Y_i)_{i\in\mathbb{Z}} \in \cdot)$ and the shift operation $T((y_i)_{i\in\mathbb{Z}}) := (y_{i+1})_{i\in\mathbb{Z}}$ there exists a stationary measure

$$\bar{\nu}(A) := \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \nu \circ T^{-i}(A), \tag{34}$$

called the stationary mean of ν[11,56]. It is convenient to suppose that probability space $(\Omega, \mathcal{J}, P)$ is rich enough to support a process $(\bar{Y}_i)_{i\in\mathbb{Z}}$ with the distribution $\bar{\nu} = P((\bar{Y}_i)_{i\in\mathbb{Z}} \in \cdot)$. Process $(\bar{Y}_i)_{i\in\mathbb{Z}}$ will be called the stationary coding of $(X_i)_{i\in\mathbb{Z}}$.

Processes $(X_i)_{i\in\mathbb{Z}}$, $(Y_i)_{i\in\mathbb{Z}}$, and $(\bar{Y}_i)_{i\in\mathbb{Z}}$ have isomorphic shift-invariant algebras for some nice coding functions, called synchronizable injections.[11] Coding function (33) is an instance of such an injection. Thus processes $(Y_i)_{i\in\mathbb{Z}}$ and $(\bar{Y}_i)_{i\in\mathbb{Z}}$ obtained from process (27) using (33) are nonergodic if $p_k = 0$ and ergodic if $p_k \in (0, 1)$.

Now let us consider block mutual information for the stationary coding $(\bar{Y}_i)_{i\in\mathbb{Z}}$ of process (27) using coding function (33). I have shown[13] that

$$\liminf_{m\to\infty} \frac{E_{\bar{\nu}}(m)}{(m \log^{-1} m)^{\beta}} > 0, \tag{35}$$

for $E_{\bar{\nu}}(m) = I_P\left(\bar{Y}_{1:m}; \bar{Y}_{m+1:2m}\right)$ and cross-over probabilities $p_k \le P(K_i = k)$. This bound should be contrasted with inequality

$$\limsup_{m\to\infty} \frac{E_{\bar{\nu}}(m)}{m^{\beta}} > 0 \tag{36}$$

which follows for $p_k = 0$.[11,12] It is an interesting open problem whether (36) can be generalized for $p_k > 0$.

Another interesting open problem concerns the question whether the stationary coding is a finite-energy process. This property is assumed in Theorems 1 and 3 to bound the length of the longest repeat and hence to bound the vocabulary size.[12] I have shown that process $(\bar{Y}_i)_{i\in\mathbb{Z}}$ is a finite-energy process for $\beta > 0.7728...$ and $p_k = 0$[11] but I wonder if it also holds for other exponents β and cross-over probabilities p_k.

IV. AFTERTHOUGHTS FOR THEORETICIANS

The translation of abstract mathematical results back into linguistic reality can be challenging. In the following, I want to share a few remarks about theoretical limitations of my constructions as models of natural language. This part of the paper is born by typical comments I receive about my model.

A. What are those ‘facts’?

Many people to whom I have presented the concept of Santa Fe processes ask the question: “What are those ‘facts’?” Whereas in models (22) and (27) the facts are just some binary variables, I tried to interpret these variables in Section II D as independent propositions about particular complicated infinite random objects, consistently described in the texts. These objects might be static like a mathematical or physical constant or might evolve slowly like cultural heritage. However, the identification of the sequence of independent facts described in the actual texts in natural language is left as a matter of future research.

This does not mean that nothing can be said about the interpretation of facts at the moment. Let me make an important remark. In my model, the probability of mentioning independent propositions in texts obeys a power law. If the same applies to natural language, it seems unlikely that the mentioned facts are the binary digits of halting probability Ω, which has an appealing property of representing a large body of mathematical knowledge in a concise form.[52,54] Although the digits of Ω have been proved to be in a sense independent (i.e., algorithmically random), I suppose that information relayed by humans in a repetitive way is mostly unrelated to Ω because human beings do not have supernatural powers to guess the bits of Ω at a power-law rate. The facts that are usually mentioned in texts should concern ‘more everyday’ objects.

B. Are facts and words the same?

Another type of reaction I have heard is: “But facts and words are the same so your result about the impli-

cation of power laws for them is a tautology!" My short answer to the criticism is this: "Words and facts are very different entities, however, so my result is nontrivial."

To support this reply let us notice the following. F. de Saussure made a famous observation that a linguistic sign is a pair of a word (i.e., a string of characters) and a meaning (roughly, an object to which the word refers).[57] To a large extent, the mapping between words and objects is one-to-one. Therefore,

$$\langle\text{number of referred objects}\rangle \approx \langle\text{number of words}\rangle\,.$$

In contrast, I have claimed a relationship

$$\langle\text{number of words}\rangle \gtrsim \frac{\langle\text{number of independent facts}\rangle}{\log\langle\text{length of text}\rangle}.$$

That inequality can be strictly sharp because objects (say, things, concepts, qualities, or activities) are different entities than facts (i.e., propositions which assume binary values). The inequality is also nontrivial because propositions usually consist of more than one word.

C. Finite active vocabulary and division of knowledge

An important limitation of my results is their asymptotic character. I have dealt with asymptotic statements because it is simpler to work out a mathematical model in that case. In reality, however, the number of different words actively used by a single person is of order $r(w) \approx 10^3 \div 10^4$. For word ranks below that value, Zipf's law (1) is observed with $\alpha \approx 1$. In contrast, for larger word ranks, word frequencies decay exponentially in collections of texts written by a single author.[29] It is not known whether a similar breakdown arises for vocabulary of admissibly minimal grammars or for Hilberg's law (10). This question is worth investigating.

Whereas word frequencies decay ultimately exponentially for single-author collections of texts, a different relationship is observed in multi-author text collections. Namely, for $r(w) \gtrsim 10^3 \div 10^4$, the exponent in the Zipf-Mandelbrot law (1) switches to $\beta \approx 0.4$ rather than to $\beta \approx 0$.[28,29] This phenomenon can be interpreted as developing social structures to maintain and transmit a larger body of knowledge than any individual could manage on his or her own. To model this phenomenon properly we should assume that finite texts produced by single authors are woven up into a discourse (a communication network) of yet unrecognized topology, rather than concatenated in an arbitrary infinite sequence $(X_i)_{i\in\mathbb{Z}}$.

D. How does language differ to maths, music, and DNA?

Texts in natural language are not the only type of a complex communication system that occurs in nature. Examples of other systems are musical transcripts, mathematical writings, computer programs, or genome (DNA and RNA). One may investigate quantitative laws obeyed in these systems, just as it is done for natural language.[58,59] Moreover, although the notion of a word is connected to linguistics, one may investigate Hilberg's conjecture and statistical properties of admissibly minimal grammars for any symbolic sequence. One may also try to interpret or predict respective experimental results theoretically.

For example, Ebeling et al. estimated n-symbol entropy by counting n-tuples in samples of texts in natural language and classical music. They confirmed formula (8) for $n \le 15$ characters with $\beta \approx 0.5$ for natural language texts and $\beta \approx 0.25$ for classical music transcripts.[4,60]

Mathematical writings are another interesting communication system which has not been much researched from a quantitative perspective. I suppose that mathematical writings obey relaxed Hilberg formula (10) similar to that of music or novels in natural language because all these symbolic sequences are produced by humans for humans, either for their work or entertainment. In either case, I suppose that humans need a large degree of repetition to learn from an information source how to react to it properly. Hence relationship (10) should arise. Intuitively, our abilities to use a particular language, to enjoy a particular style of music, or to work in a branch of mathematics are all learned and learning is only possible if there are some patterns to be learned.

In contrast, computer programs or DNA are sequences that control behavior of machines like computers or biological cells. These machines can interpret control sequences in a fixed manner without learning or loss of synchronization caused by other factors. Hence there is less need of repetition in the control sequences. Consequently, relationship (10) and word-like structures need not arise in compiled computer programs or DNA to such an extent as in typical texts in natural language.

V. AFTERTHOUGHTS FOR EXPERIMENTALISTS

The body of theoretical insight gathered so far asks for experimental verification. One can rightly question to what extent these results may be applied to real texts in natural language. The preparation of a sound experimental study requires much more space than it is left in this paper, so let me only sketch a few problems. There are several levels of experimental verification, which correspond to growing difficulty.

The easiest thing to do is to check whether Zipf's or Herdan's law is satisfied for languages in which words are delimited by spaces. There are plenty of articles about that, including observations that Zipf's law breaks for large ranks.[28,29] In contrast, it is a bit harder to verify whether a power-law is satisfied for nonterminals in admissibly minimal grammars. The next task is to verify Hilberg's conjecture for a particular text. In the end, the hardest thing to do is to estimate how many facts are jointly described in two given texts.

In this section, I will touch some of these questions. First, I discuss some grammar transforms that *may not* be used to approximate admissibly minimal grammars or to detect word boundaries. Second, I suggest how mutual information can be efficiently estimated.

A. What is the appropriate grammar-based code?

I have claimed that there is a tight relationship between the number of distinct words in a text in natural language and the number of distinct nonterminal symbols in an admissibly minimal grammar for the text. Although this claim is supported by several computational experiments,[39,42,43] the regression between these two quantities has not been surveyed directly so far. In fact, investigating this dependence is hard because computing admissibly minimal grammars is extremely costly, even in approximation.[42,43] In contrast, computationally less intensive grammar transforms may detect spurious structures.

For example, irreducible grammar transforms[40,44,45] exhibit a power-law growth of vocabulary size for any source with a positive entropy rate.[41] To see it, let us first observe inequality

$$|\mathsf{G}| - \mathbf{V}[\mathsf{G}] \le (\mathbf{V}[\mathsf{G}] + \operatorname{card}\mathbb{X})^2, \tag{37}$$

where G is an irreducible grammar[44] and $|\mathsf{G}|$ is the Yang-Kieffer length of G.[12] Any irreducible grammar satisfies (37) since any concatenation of two symbols may only occur on the right-hand sides of its rules only once.

What happens if an irreducible grammar G compresses a text of length n produced by a stationary source with entropy rate h_μ? Then we obtain $|\mathsf{G}| \gtrsim h_\mu n/\log n$ from the source coding inequality $|\mathsf{G}|\log|\mathsf{G}| \gtrsim h_\mu n$ and the trivial inequality $|\mathsf{G}| \le n$. Combining that with (37) yields the power law

$$\mathbf{V}[\mathsf{G}] \gtrsim \sqrt{\frac{h_\mu n}{\log n}} - \operatorname{card}\mathbb{X} - 1. \tag{38}$$

In particular, the higher the entropy rate is, the more nonterminals are detected by the grammar.

In my opinion relationship (38) is an artifact. It arises because irreducible grammars minimize a wrongly chosen length function. If we choose a certain different grammar length function[12,42] then, after complete minimization, we obtain admissibly minimal grammars. The number of nonterminals in approximations of such grammars is a thousand times larger for texts in natural language than for IID sources.[41] Thus I suppose that the vocabulary size of admissibly minimal grammars is lower-bounded by the n-symbol excess entropy rather than by the entropy rate.

B. How to measure mutual information?

Entropy of a long sequence of random variables is hard to estimate. It can be effectively bounded only from above and there is some systematic nonnegligible error term. We know that an upper bound for the entropy of a text is given by the expected length of any prefix-free code for the text. However, the length of the shortest effectively decodable code for the text equals the algorithmic complexity of the text, which is greater than the entropy. Thus any intention of estimating Shannon entropy by universal coding ends up with estimating algorithmic complexity.

Now, certain care must be given to distinguishing Shannon entropy $H_P(X_1^n)$ and algorithmic complexity $H(X_1^n)$. Although we have inequality (11) for any computable measure P, the difference

$$\mathbf{E}_P\, H(X_1^n) - H_P(X_1^n) \tag{39}$$

can exceed any sublinear function of n if P is stationary but not computable.[61]

Whereas the classical proof of unboundedness of (39) is difficult,[61] a similar result can be obtained using Santa Fe process (22). Let P be the probability measure for process (22) and let

$$F = P(\,\cdot\,|(Z_k)_{k\in\mathbb{N}}) \tag{40}$$

be its conditional measure.[10,62] The values of conditional measure F depend on the value of process $(Z_k)_{k\in\mathbb{N}}$. In particular, measure F is not computable for algorithmically random $(Z_k)_{k\in\mathbb{N}}$ (i.e., $F(X_1^n)$ cannot be computed given no $(Z_k)_{k\in\mathbb{N}}$). Further,

$$\begin{aligned} H_P(X_1^n) - \mathbf{E}_P\, H_F(X_1^n) &= I_P(X_1^n;(Z_k)_{k\in\mathbb{N}}) \\ &= O(n^\beta). \end{aligned} \tag{41}$$

Moreover, we have the source coding inequality $H_P(X_1^n) \le \mathbf{E}_P\, H(X_1^n) = \mathbf{E}_P\,\mathbf{E}_F\, H(X_1^n)$. Hence we obtain

$$\mathbf{E}_P\left[\mathbf{E}_F\, H(X_1^n) - H_F(X_1^n)\right] \ge O(n^\beta) \tag{42}$$

as the desired result, where an analogue of (39) appears. In other words, universal coding bounds suffer from a large systematic error for Shannon entropy of noncomputable probability measures.

Now I will show that the error of the coding bounds can be greatly reduced for certain noncomputable measures when we rather bound algorithmic complexity. This opens way to bounding also algorithmic information, which is a difference of complexities.

Suppose that

$$P = \int P(\,\cdot\,|\Theta)dP \tag{43}$$

where $P(\,\cdot\,|\Theta)$ are measures of stationary Markov chains for particular values of transition probabilities Θ and $P(\Theta \in \cdot\,)$ is an appropriate prior over all transition probabilities for all possible orders of Markov chains. Measures $P(\,\cdot\,|\Theta)$ are not computable for algorithmically random Θ. It is likely, however, that the Shannon-Fano code

yielded by measure (43) is computable and universal, i.e., $-\log P(X_1^n)$ can be computed given X_1^n and

$$\lim_{n\to\infty} \frac{1}{n} \mathbf{E}_Q \left[-\log P(X_1^n)\right] = h_\nu \tag{44}$$

holds for any stationary measure $\nu = Q((X_i)_{i\in\mathbb{Z}} \in \cdot)$.

Consider now pointwise mutual information

$$\begin{aligned} &I^P(x_1^n; x_{n+1}^{2n}) := \\ &\quad H^P(x_1^n) + H^P(x_{n+1}^{2n}) - H^P(x_1^{2n}), \end{aligned} \tag{45}$$

using pointwise entropy

$$H^P(x_1^n) := -\log P(X_1^n = x_1^n). \tag{46}$$

The Shannon-Fano coding yields inequality

$$H(x_1^n) \le H^P(x_1^n) + C_n^P \tag{47}$$

where x_1^n is arbitrary and $C_n^P = c_P + 2\log n$ for a certain constant c_P. Thus we define the loss of pointwise mutual information with respect to algorithmic information as

$$L^P(x_1^n; x_{n+1}^{2n}) := I^P(x_1^n; x_{n+1}^{2n}) - I(x_1^n; x_{n+1}^{2n}). \tag{48}$$

We will next use the following lemma:

Lemma 1 *Consider a function $G : \mathbb{N} \to \mathbb{R}$ such that $\lim_k G(k)/k = 0$ and $G(n) \ge 0$ for all but finitely many n. Then for infinitely many n, we have $2G(n) - G(2n) \ge 0$.*[12]

If (44) holds indeed then equality

$$\lim_{n\to\infty} \frac{1}{n} \mathbf{E}_Q\, H(X_1^n) = h_\nu \tag{49}$$

for any stationary measure Q and Lemma 1 for function $G(n) = \mathbf{E}_Q \left[H^P(X_1^n) - H(X_1^n)\right] + C_n^P$ yield

$$\limsup_{n\to\infty} \mathbf{E}_Q \left[L^P(X_1^n; X_{n+1}^{2n}) + C_n^P\right] \ge 2\log 2. \tag{50}$$

Hence, as long as (44) holds, pointwise mutual information (45) is an upper estimate of algorithmic information, up to a small logarithmic correction C_n^P.

In certain cases of noncomputable Q, quantity (45) is also a lower estimate of algorithmic information. Observe that P is computable. Hence for all P-algorithmically random sequences $(x_i)_{i\in\mathbb{N}}$ we have by definition

$$\inf_{n\in\mathbb{N}} \left[H(x_1^n) + \log P(x_1^n)\right] > -\infty. \tag{51}$$

Moreover, we have the following fact:

Theorem 4 *The set of P-algorithmically random sequences is the union of sets of $P(\cdot|\Theta)$-algorithmically random sequences over all parameters Θ that are algorithmically random against the prior $P(\Theta \in \cdot)$.*[63–65]

Each of those sets of algorithmically random sequences has the respective full measure, so in a sense it contains all outcomes typical of that measure. Let us fix a sequence $(x_i)_{i\in\mathbb{N}}$ belonging to one of these sets. Because P is stationary, by (47) and (51), we obtain

$$\sup_{n\in\mathbb{N}} \left[L^P(x_1^n; x_{n+1}^{2n}) - 2C_n^P\right] < \infty. \tag{52}$$

Hence, inequality (52) gives an upper bound for loss (48) for typical realizations of typical Markov chains.

In view of bounds (50) and (52), pointwise mutual information (45) could be considered an interesting estimate of algorithmic information. It could be used for verifying (or rather falsifying) relaxed algorithmic Hilberg conjecture (10). The details of computing distribution P and pointwise mutual information (45) will be worked out in another paper, however. Although the sketched distribution P is computable, there are some problems with assuring *efficient* computability.

VI. CONCLUSION

In this article I have presented a wide array of interesting issues that arise when quantitative research in language is combined with fundamental research in information theory. As I have indicated in the Introduction, the presented ideas may be inspiring for more general studies of complex systems because of the demonstrated connections among excess entropy, power laws, and the emergence of hierarchical structures in data.

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